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On the Multi-Parameter Characteristic Perturbation Method; Application to Nonlinear Supersonic Nonequilibrium Flow over a Wedge

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SUMMARY
The method of strained coordinates is extended to expand the dependent variables as well as independent coordinates of a nonlinear hyperbolic system in asymptotic series of several parameters. The perturbation parameters may be of a different nature but are required to be intrinsically independent of each other. The method is found to be particularly useful for treating problems with several relevant parameters being of the same order of one another. The illustrative example discussed is a nonlinear supersonic nonequilibrium flow over a wedge where the nonlinear effect in the flow becomes of the same order as the nonequilibrium effect. A second-order theory is developed to provide a description of the near-field flow pattern and the expression of the front frozen shock wave attached to the nose of the wedge.

1. General Remarks

The perturbation method of Poincaré [1], which consists of developing the solution of an initial or boundary value problem in an asymptotic series of a parameter which appears explicitly either in the problem or is introduced artificially, is one of the standard analytical methods of solution of nonlinear problems of applied mechanics and physics. In Poincaré's early work, as well as in the subsequent development of his ideas, one may find some interesting germs for conceivable further generalizations [2]. One of the important developments in the singular perturbation scheme, namely the method of strained coordinates [3], stems originally from Poincaré's periodic solutions of nonlinear ordinary differential equations by the straining of the independent coordinates. Lighthill [4] later suggested a general technique for removing nonuniformities from perturbation solutions of nonlinear problems. The principle of Lighthill's technique is that the linearized solution may have the right form, but not at the right place. The discrepancy is removed by straining one of the independent coordinates, i.e., the chosen independent coordinate is expanded in an asymptotic series as are the dependent variables. Lin [5] further advances the technique for hyperbolic equations in two variables by adopting characteristic parameters as the basis for a perturbation theory, which amounts to straining both families of characteristics. This singular perturbation technique has proved altogether successful for treating nonlinear hyperbolic systems [6], [7], [8]. Further generalization and systematization of the method may be found in the work of Chou and Chu [9] in treating rather complicated nonlinear nonequilibrium flow problems.

Another interesting generalization from Poincaré's original idea is the so-called multi-parameter perturbation technique suggested by Nowinski and Ismail in treating some elastostatic problems [10]. The ordinary perturbation technique is extended to embrace the expansions of the desired quantities in powers of several parameters. The parameters involved may be of a different character: some, for example, describing the material properties, some describing the dynamic or geometrical nature of the problems, and so on. However, parameters concerned in the perturbation scheme must be intrinsically independent of each other.

The present investigation is concerned with combining the previously mentioned techniques together to treat a nonlinear supersonic nonequilibrium flow over a wedge. The combining method may be illustrated as follows: Suppose the boundary value problem under consideration includes N parameters ε_i ($i=1, 2, ..., N$) intrinsically independent of each other, and the independent variables are x and y. A pair of characteristics labelled as α and β is properly chosen to form the new independent coordinates and the system of governing equations and boundary conditions is then transformed from the xy-plane into the $\alpha\beta$-plane. Solutions $f_v(\alpha, \beta, \varepsilon)$ may be constructed in powers of N parameters as

$$f_v(\alpha, \beta; \varepsilon) = \sum_{m,n,\dots,p=0}^{\infty} \sum_{i,j,\dots,l=1}^{N} (\varepsilon_i)^m (\varepsilon_j)^n \dots (\varepsilon_l)^p \cdot \phi_{i,j,\dots,l}^{m,n,\dots,p}(\alpha, \beta),$$

where f_v represents the dependent variables including the new pair of dependent variables x and y, and all the boundary conditions of the system should be expanded accordingly.

To each successive order $K=m+n+\dots+p$ one obtains an associated set of boundary value subsystems, which may then be solved by the standard operational methods. The details of coordinate transformations and multi-parameter perturbations will be presented in the later sections.

The illustrative problem selected here is a nonlinear supersonic nonequilibrium flow over a wedge. The numerous linearized analyses of the corresponding problem [11], [12], [13], and [14] describe the so-called nonequilibrium effect (or relaxation effect) for which the flow properties exhibit an exponential decay with respect to the distance away from the disturbance. In other words, the dissipative nonequilibrium effect tends to smooth out the abrupt compression generated by the wedge. However, the well-known nonlinear amplitude dispersion effect, which tends to steepen the compression gradients, has not been taken into account in the linearized analyses. Linearized theory fails to give any information about the position and strength of the shock wave, nor does it give a satisfactory description of the flow field at large distances from the disturbance. Besides the numerical analysis [15] of the problem, some nonlinear analyses of the problem may be found in the case of near-equilibrium ($\sigma \ll \varepsilon$), [16], and in the case of relaxation-dominated ($\varepsilon \ll \sigma$) flow [17], where the parameter σ characterizes the nonequilibrium effect and the parameter ε characterizes the nonlinear effect. The problem examined here is concerned with the fact that the nonequilibrium effect is of the same order as the nonlinear effect [$\sigma = O(\varepsilon)$]. The system of governing equations is solved by employing the multi-parameter characteristic perturbation method. Carrying the calculation to the second-order of the semi-nose wedge angle, the front frozen shock wave is constructed from the solution of the fluid properties and the relations obtained by applying a similar perturbation scheme to the shock jump conditions. However, it is well-known that at large distances from the disturbance, the nonequilibrium process must eventually proceed toward equilibrium ($\varepsilon \gg \sigma$). Consequently, the present second-order theory should only be regarded as a near-field solution.

2. Description of the Problem

The basic equations governing the motion of a steady nonequilibrium supersonic flow over a wedge can be easily reduced from the governing equations of a general nonequilibrium flow system (see, for example, the book of Vincenti and Kruger [18], or Clarke and McChesney [19]). They are

$$\begin{aligned}
&\rho u u_x + \rho v u_y + p_x = 0\,,\\
&\rho u v_x + \rho v v_y + p_y = 0\,,\\
&\frac{1}{a_f^2}(u p_x + v p_y) + \rho u_x + \rho v_y = \frac{h_q}{h_\rho}\dot{q}(p, \rho, q)\,,\\
&u q_x + v q_y = \dot{q}(p, \rho, q)\,,\\
&\frac{1}{a_f^2}(u p_x + v p_y) - (u \rho_x + v \rho_y) = \frac{h_q}{h_\rho}\dot{q}(p, \rho, q)\,,\\
&h = h(p, \rho, q)\,.
\end{aligned} \tag{2.1}$$

Here, only one nonequilibrium mode is considered, and the various transport effects are neglected. Also, x and y are the Cartesian coordinates; u and v denote respectively the velocity components along the x- and y-axis; p, ρ, h, q, and $\dot{q}$ are respectively the pressure, density, specific enthalpy, a variable characterizing the progress of the nonequilibrium process and the reaction rate, $\dot{q}=\dot{q}(p, \rho, q)$, which describes the rate of change of the progress variable q, is a known function of p, ρ, and q. Finally, the subscripts x, y, q, ρ signify partial differentiation with respect to x, y, q and ρ respectively; and the frozen speed of sound [18] is given by

$$a_f^2 = \left(\frac{\partial p}{\partial \rho}\right)_{s.q} = \frac{-h_\rho}{h_p - 1/\rho} . \tag{2.2}$$

When the frozen Mach number $M_f = [(u^2+v^2)/a_f^2]^{\frac{1}{2}}$ is greater than one, the system (2.1) is hyperbolic and possesses three families of real characteristics, namely, the outgoing and incoming Mach waves, and the streamlines, i.e.,

$$\begin{aligned} \lambda_{1,2} &= \frac{dx}{dy} = \frac{-uv \pm a_f(u^2+v^2-a_f^2)^{\frac{1}{2}}}{a_f^2 - v^2} = \cot(\theta \pm \mu_f) , \\ \lambda_3 &= \frac{dx}{dy} = \frac{u}{v} = \cot\theta , \end{aligned} \tag{2.3}$$

where $\theta = \tan^{-1}(v/u)$ being the flow angle and μ_f being the frozen Mach angle.

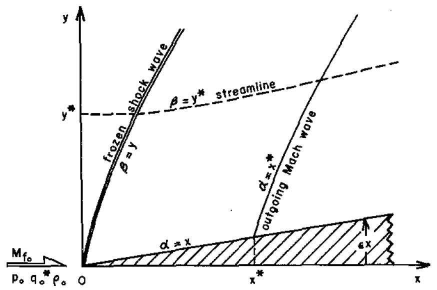

Figure 1. Labelling of the characteristic coordinates.

A characteristic coordinate system (α, β) may be introduced as follows: α is constant along an outgoing Mach line such that if this line intersects the surface of the wedge at a point $x = x^*$, the line will be labelled as $\alpha = x^*$; β is constant along a streamline such that if this line intersects the front shock wave at a point $y = y^*$, the line will be labelled as $\beta = y^*$ (see Figure 1). The transformation relationships between (x, y) on one hand and (α, β) on the other can be deduced immediately from $dx = x_\alpha d\alpha + x_\beta d\beta$ and $dy = y_\alpha d\alpha + y_\beta d\beta$. In terms of α and β, Eqs. (2.1) and (2.3) become

$$\begin{aligned} & x_\alpha = u y_\alpha / v , \\ & x_\beta = \lambda y_\beta \\ & \rho u u_\alpha + \rho v v_\alpha + p_\alpha = 0 , \\ & u q_\alpha = \dot{q} x_\alpha , \\ & \rho v u_\beta - \rho u v_\beta - [uv + \lambda(a_f^2 - v^2) p_\beta]/a_f^2 + (u - \lambda v)\dot{q} y_\beta h_q / h_p = 0 , \\ & \rho u(v_\alpha y_\beta - v_\beta y_\alpha) + \rho v(x_\alpha v_\beta - v_\alpha x_\beta) + (x_\alpha p_\beta - p_\alpha x_\beta) = 0 , \\ & p_\alpha / a_f^2 - \rho_\alpha - q_\alpha h_q / h_p = 0 , \end{aligned} \tag{2.4}$$

where λ is the λ_1 appearing in (2.3) i.e., the direction of the outgoing Mach waves.

For later convenience, the equilibrium speed of sound a_e is introduced as follows:

$$a_e^2 = \left(\frac{\partial p}{\partial \rho}\right)_{s,q^*} = -\frac{h_\rho + q_\rho^*}{h_p + h_q q_p^* - 1/\rho}, \tag{2.5}$$

where q^* is the local equilibrium value of q.

Let the wedge be described by the equation $y = \varepsilon x$ where ε is the tangent of the semi-nose angle. The condition that the flow be tangent to the wedge implies that $v/u \to \varepsilon$ as $y \to \varepsilon x$. In the $\alpha\beta$ plane, one has

$$\begin{aligned} &\frac{v}{u} \to \varepsilon, \text{ as } \beta = 0, \\ &x = \alpha, \text{ at } \beta = 0 \\ &y = \beta, \text{ at front shock}. \end{aligned} \tag{2.6}$$

The last two conditions are the consequence of the labelling of α, β-coordinates. In addition, the usual jump condition must be satisfied at each point on the nose frozen shock. Consequently, if δ is frozen shock angle and subscript "0" denotes the free-stream condition, which is assumed to be in the thermodynamic equilibrium, the jump conditions are

$$\begin{aligned} &\rho(u - v \cot \delta) = \rho_0 u_0, \quad h + \tfrac{1}{2}u^2 = h_0 + \tfrac{1}{2}u_0^2 \\ &p - p_0 = \rho_0 u_0 (u_0 - u), \quad v = (u_0 - u) \cot \delta, \ q = q_0, \end{aligned} \tag{2.7}$$

respectively the continuity equation, the energy equation, the momentum balance in the normal and tangential directions, and the continuity of q. The position of the front shock is, of course, not known. However, it must assume such a form that $dx/dy = \cot \delta$ at every point on the frozen shock. The front shock wave will be determined later.

3. Multi-Parameter Characteristic Perturbation

The present investigation is concerned with the nonequilibrium flow pattern of the wedge in the case that one of the fluid parameters $\sigma = a_{f0}^2/a_{e0}^2 - 1$ is small and is of the same order as the geometric property ε, i.e., $\sigma = O(\varepsilon)$. Since both ε and σ are considered small in the present case, and they are also intrinsically independent of each other, the solution of the system may be assumed to be constructed in a power series of parameters ε and σ in the $\alpha\beta$-plane as follows:

$$w(\alpha, \beta) = w_0(\alpha, \beta) + \varepsilon w_1(\alpha, \beta) + \sigma W_1(\alpha, \beta) + \varepsilon^2 w_2(\alpha, \beta) + \varepsilon\sigma \bar{w}(\alpha, \beta) + \sigma^2 W_2(\alpha, \beta) + \ldots, \tag{3.1}$$

where w can be any dependent variable which appears in the system (2.4) and w_0 represents the value of w in the undisturbed region. In particular $v_0 = 0$.

Making use of Taylor's series, λ may be expanded as

$$\lambda = \lambda_0 + \varepsilon\lambda_1(\alpha, \beta) + \sigma\lambda_1^\sigma(\alpha, \beta) + \varepsilon^2 \lambda_2(\alpha, \beta) + \ldots,$$

where

$$\begin{aligned} &\lambda_0 = (M_0^2 - 1)^{\frac{1}{2}}, \\ &\lambda_1 = -\frac{M_{f0}^2}{u_0} v_1(\alpha, \beta) - \frac{M_{f0}^2}{\lambda_0 u_0} \kappa_1 p_1(\alpha, \beta) - \frac{M_{f0}^4}{2\lambda_0 u_0^2} \kappa_2 q_1(\alpha, \beta), \end{aligned} \tag{3.2}$$

etc., and

$$\kappa_1 = \left[\frac{1}{\rho} + \frac{M_f^2}{2}\left(\frac{\partial a_f^2}{\partial p}\right) + \frac{M_f^2}{2a_f^2}\left(\frac{\partial a_f^2}{\partial \rho}\right)\right]_0,$$

$$\kappa_2 = \left[\frac{\partial a_f^2}{\partial q} - \frac{h_q}{h_\rho}\frac{\partial a_f^2}{\partial \rho}\right]_0,$$

where subscript "0" appended to a bracket signifies that all quantities in the bracket are evaluated at the undisturbed state of the oncoming stream.

Similarly, the expansion of $\dot{q}(p, \rho, q)$ is

$$\dot{q}(p,\rho,q) = \frac{1}{\tau_0}\left[\varepsilon(-q_1)+\sigma(-Q_2)+\varepsilon^2(-q_2-\eta p_1^2) + \varepsilon\sigma\left(-q_{12}-\frac{h_{\rho 0}}{h_{q0}}\frac{p_1}{a_{f0}^2}\right)+\sigma^2\left(-Q_2-\frac{h_{\rho 0}}{h_{q0}}\frac{p_1}{a_{f0}^2}\right)+\dots\right], \tag{3.3}$$

where

$$\tau_0 = -h_{\rho 0}/[\dot{q}_q(h_\rho+h_q q_\rho^*)]_0, \text{ and}$$

$$\eta = (q_{pp}+2q_{p\rho}/a_f^2+q_{\rho\rho}/a_f^4)_0/2(1+h_q q_\rho/h_\rho)_0 .$$

Substituting Eqs. (3.1), (3.2) and (3.3) into system (2.4) and collecting terms of like order of ε, σ, ε^2, $\varepsilon\sigma$, σ^2, ..., the zeroth-order and the first-order governing equations are found to be

$$\varepsilon^0: \quad y_{0\alpha}=0 \qquad x_{0\beta}=\lambda_0 y_{0\beta} \tag{3.4}$$

$$\begin{aligned}
\varepsilon^1: \quad & x_{0\alpha}v_1 = u_1 y_{0\alpha}+u_0 y_{1\alpha}, \quad x_{1\beta}=\lambda_1 y_{0\beta}+\lambda_0 y_{1\beta},\\
& \rho u_0 u_{1\alpha}+p_{1\alpha}=0, \quad \tau_0 u_0 q_{1\alpha}+x_{0\alpha}q_1=0,\\
& \rho_0 u_0 v_{1\beta}+(h_{q0}/h_{\rho 0})u_0 y_{0\beta}q_1/\tau_0+\lambda_0 p_{1\beta}=0,\\
& \rho_0 u_0 y_{0\beta}v_{1\alpha}+x_{0\alpha}p_{1\beta}-x_{0\beta}p_{1\alpha}=0,\\
& p_{1\alpha}/a_{f0}^2-\rho_{1\alpha}-(h_{q0}/h_{\rho 0})u_0 q_1/\tau_0=0.
\end{aligned} \tag{3.5}$$

Equations for the other order systems will be discussed in the subsequent sections.

In a similar manner, the boundary conditions of the system can be decomposed into a set of conditions, according to different powers of ε, σ, etc.

As pointed out previously the position of the front shock wave is not known *a priori*. However, it is understood that the shock would decay into a Mach line $\alpha=0$ if the disturbance approaches zero, i.e. $\varepsilon\to 0$. It is assumed that the frozen shock wave expression in $\alpha\beta$-plane may be of the following form:

$$\alpha=\alpha(\beta,\varepsilon,\sigma)=\varepsilon f_1(\beta)+\sigma F_1(\beta)+\varepsilon^2 f_2(\beta)+\varepsilon\sigma\bar{f}(\beta)+\sigma^2 F_2(\beta)+\dots \tag{3.6}$$

Functions f's and F's will be determined later.

From the perturbation of relevant boundary conditions, we have the following conditions:

$$\varepsilon^0: \quad x_0=0, \text{ at } \beta=0; \quad y_0=\beta, \text{ at } \alpha=0. \tag{3.7}$$

$$\left.\begin{aligned}
\varepsilon^1: \quad & x_1=0, \text{ at } \beta=0; \quad y_1=0, \text{ at } \alpha=0.\\
& v_1=u_0, \text{ at } \beta=0; \quad q_1=0, \text{ at } \alpha=0.
\end{aligned}\right\} \tag{3.8}$$

$$\left.\begin{aligned}
\sigma^1: \quad & X_1=0, \text{ at } \beta=0; \quad Y_1=0, \text{ at } \alpha=0,\\
& V_1=0, \text{ at } \beta=0; \quad Q_1=0, \text{ at } \alpha=0.
\end{aligned}\right\} \tag{3.9}$$

$$\left.\begin{aligned}
\varepsilon^2: \quad & x_2=0, \text{ at } \beta=0; \quad y_2=0, \text{ at } \alpha=0,\\
& v_2=u_1, \text{ at } \beta=0; \quad q_2=0, \text{ at } \alpha=0.
\end{aligned}\right\} \tag{3.10}$$

$$\left.\begin{aligned}
\varepsilon^1\sigma^1: \quad & \bar{x}=0, \text{ at } \beta=0; \quad \bar{y}=0, \text{ at } \alpha=0,\\
& \bar{v}=U_1, \text{ at } \beta=0; \quad \bar{q}=0, \text{ at } \alpha=0.
\end{aligned}\right\} \tag{3.11}$$

$$\left.\begin{aligned}
\sigma^2: \quad & X_2=0, \text{ at } \beta=0; \quad Y_2=0, \text{ at } \alpha=0,\\
& V_2=0, \text{ at } \beta=0; \quad Q_2=0, \text{ at } \alpha=0, \text{ etc.}
\end{aligned}\right\} \quad 3.12)$$

4. Solution of the Problem

The solutions of the zeroth order system (3.4) and (3.7) are easily found to be

$$\varepsilon^0: \quad x_0 = \alpha + \lambda_0 \beta, \qquad y_0 = \beta.$$

The first-order system (3.5) together with boundary condition (3.8) can be solved by the Laplace transform method (a summary of this method is included in Appendix A). They are

$$\begin{aligned}\varepsilon^1: \quad & v_1 = u_0, \quad p_1 = \rho_0 u_0/\lambda_0, \quad u_1 = -u_0/\lambda_0, \\ & \rho_1 = M_{f0}^2 \rho_0/\lambda_0, \quad q_1 = 0, \quad y_1 = \alpha, \\ & x_1 = -M_{f0}^2 - \kappa_1 M_{f0}^4 \rho_0 u_0 \beta/(2\lambda_0^2),\end{aligned} \tag{4.2}$$

where κ_1 is given in Eq. (3.2).

In the σ-order system, X_1, Y_1, U_1, V_1, P_1, R_1, and Q_1, satisfy the same set of differential equations as x_1, y_1, u_1, v_1, p_1, ρ_1 and q_1. However, there are no inhomogeneous terms in the system ($V_1=0$), consequently, one has

$$X_1=0,\ Y_1=0,\ U_1=0,\ V_1=0,\ P_1=0,\ R_1=0,\ Q_1=0. \tag{4.3}$$

Governing equations for ε^2-order system are

$$\begin{aligned}\varepsilon^2: \quad & v_2 = u_0 y_{2\alpha} + u_1, \quad x_{2\beta} = \lambda_2 + \lambda_0 y_{2\beta}, \\ & \rho_0 u_0 u_{2\alpha} + p_{2\alpha} = 0, \quad \tau_0 u_0 q_{2\alpha} = -q_2 - \eta p_1^2, \\ & \rho_0 u_0 v_{2\beta} + \lambda_0 p_{2\beta} = -(h_{q0} u_0)/(h_{\rho 0}\tau_0)(q_2 + \eta p_1^2), \\ & \rho_0 u_0 v_{2\alpha} + p_{2\beta} - \lambda_0 p_{2\alpha} = 0, \\ & p_{2\alpha}/a_{f0}^2 - \rho_{2\alpha} - q_{2\alpha} h_{q0}/h_{\rho 0} = 0,\end{aligned} \tag{4.4}$$

where λ_2 has the same form of λ_1.

The solutions for this inhomogeneous system and (3.10) are [see Appendix A]

$$\begin{aligned}\varepsilon^2: \quad v_2(\alpha,\beta) &= -\frac{u_0}{\lambda_0} - \tfrac{1}{2}\frac{h_{q0}}{h_{\rho 0}}\frac{\rho_0 u_0^4}{\tau_0 \lambda_0^2}\eta\beta \exp\left(-\frac{\alpha}{u_0\tau_0}\right), \\ p_2(\alpha,\beta) &= -\tfrac{1}{2}\frac{h_{q0}}{h_{\rho 0}}\frac{\rho_0^2 u_0^5}{\lambda_0^3 \tau_0}\eta\beta \exp\left(-\frac{\alpha}{u_0\tau_0}\right), \\ q_2(\alpha,\beta) &= \frac{\rho_0^2 u_0^4}{\lambda_0^2}\left[1 - \exp\left(-\frac{\alpha}{u_0\tau_0}\right)\right], \\ u_2 &= -\frac{1}{\rho_0 u_0} p_2(\alpha,\beta), \quad \rho_2 = \frac{1}{a_{f0}^2} p_2 - \frac{h_{q0}}{h_{\rho 0}} q_2, \\ y_2(\alpha,\beta) &= \tfrac{1}{2}\frac{h_{q0}}{h_{\rho 0}}\frac{\rho_0 u_0^4}{\lambda_0^2}\eta\beta\left[\exp\left(-\frac{\alpha}{u_0\tau_0}\right) - 1\right], \\ x_2(\alpha,\beta) &= \lambda_0 y_2 + \frac{M_{f0}^2}{\lambda_0}\frac{u_2}{u_0}\beta + \frac{M_{f0}^2}{4}\frac{h_{q0}}{h_{\rho 0}}\frac{\rho_0 u_0^4}{u_0\tau_0\lambda_0^2}\eta\beta^2 \exp\left(-\frac{\alpha}{u_0\tau_0}\right) \\ &\quad + \kappa_1 \frac{M_{f0}^2}{4}\frac{h_{q0}}{h_{\rho 0}}\frac{\rho_0^2 u_0^4}{\lambda_0^4 \tau_0}\eta\beta^2 \exp\left(-\frac{\alpha}{u_0\tau_0}\right) + \kappa_2 \frac{M_{f0}^4 \rho_0^2 u_0^2}{2\lambda_0^3}\left[1 - \exp\left(-\frac{\alpha}{u_0\tau_0}\right)\right]_\rho \beta.\end{aligned} \tag{4.5}$$

Similarly, the governing equations for the $\varepsilon\sigma$-order system are

$$\bar{v}=u_0\bar{y}_\alpha,\quad \bar{x}_\beta=\bar{\lambda}_2+\lambda_0\bar{y}_\beta,\quad \rho_0u_0\bar{u}_\alpha+\bar{p}_\alpha=0\,,$$
$$\tau_0u_0\bar{q}_\alpha+\bar{q}=-(h_{\rho0}/h_{q0})p_1,\quad \rho_0u_0\bar{v}_\alpha+\bar{p}_\beta-\lambda_p\bar{p}_\alpha=0\,, \tag{4.6}$$
$$\rho_0u_0\bar{v}_\alpha+\bar{p}_\beta+(h_{q0}/h_{\rho0})u_0\bar{q}/\tau_0=-(u_0/\tau_0)p_1/a_{f0}^2\,,$$
$$\bar{p}_\alpha/a_{f0}^2-\bar{\rho}_\alpha-(h_{q0}/h_{\rho0})\bar{q}_\alpha=0\,.$$

The relevant boundary conditions for the system are given in (3.1).

Again, by employing the standard operational method, the solutions for the inhomogeneous system are

$$\bar{v}=-\tfrac{1}{2}\frac{M_{f0}^2}{\lambda_0\tau_0}\beta\exp\left(-\frac{\alpha}{u_0\tau_0}\right),\quad \bar{p}=-\tfrac{1}{2}\frac{\rho_0u_0M_{f0}^2}{\lambda_0^2\tau_0}\beta\exp\left(-\frac{\alpha}{u_0\tau_0}\right),$$

$$\bar{q}=\frac{h_{\rho0}}{h_{q0}}\frac{\rho_0M_{f0}^2}{\lambda_0}\left[\exp\left(-\frac{\alpha}{u_0\tau_0}\right)-1\right],\quad \bar{u}=-\bar{p}/(\rho_0u_0)\,,$$

$$\bar{\rho}=\bar{p}/a_{f0}^2-(h_{q0}/h_{\rho0})\bar{q}\,, \tag{4.7}$$

$$\bar{y}=\frac{M_{f0}^2}{2\lambda_0}\beta\exp\left(-\frac{\alpha}{u_0\tau_0}\right),$$

$$\bar{x}=\lambda_0\bar{y}+\tfrac{1}{4}\frac{M_{f0}^4}{\lambda_0\tau_0u_0}\beta^2\exp\left(-\frac{\alpha}{u_0\tau_0}\right)+\frac{\kappa_1}{4}\frac{M_{f0}^4}{\lambda_0^3\tau_0u_0}\beta^2\exp\left(-\frac{\alpha}{u_0\tau_0}\right)$$
$$+\frac{h_{\rho0}}{h_{q0}}\frac{\rho_0M_{f0}^6}{2\lambda_0^2u_0^2}\kappa_2\beta\left[\exp\left(-\frac{\alpha}{u_0\tau_0}\right)-1\right].$$

There are no inhomogeneous terms in the σ^2 system, consequently, one has

$$u_2=0,\ v_2=0,\ p_2=0,\ R_2=0,\ x_2=0\ \text{and}\ y_2=0\,. \tag{4.8}$$

Finally, the complete solutions for the second-order theory are

$$v(\alpha,\beta)=\varepsilon u_0+\varepsilon^2\left[-\frac{u_0}{\lambda_0}-\tfrac{1}{2}\frac{h_{q0}}{h_{\rho0}}\frac{\rho_0u_0^4}{\tau_0\lambda_0^2}\eta\beta\exp\left(-\frac{\alpha}{u_0\tau_0}\right)\right]$$
$$+\varepsilon\sigma\left[-\tfrac{1}{2}\frac{M_{f0}^2}{\lambda_0\tau_0}\beta\exp\left(-\frac{\alpha}{u_0\tau_0}\right)\right]+\dots,$$

$$p(\alpha,\beta)=p_0+\varepsilon\frac{\rho_0u_0^2}{\lambda_0}+\varepsilon^2\left[-\tfrac{1}{2}\frac{h_{q0}}{h_{\rho0}}\frac{\rho_0^2u_0^5}{\lambda_0^3\tau_0}\eta\beta\exp\left(-\frac{\alpha}{u_0\tau_0}\right)\right]$$
$$+\varepsilon\sigma\left[-\tfrac{1}{2}\frac{\rho_0u_0M_{f0}^2}{\lambda_0^2\tau_0}\beta\exp\left(-\frac{\alpha}{u_0\tau_0}\right)\right]+\dots,$$

$$q(\alpha,\beta)=q_0+\varepsilon^2\frac{\rho_0u_0^4}{\lambda_0^2}\left[1-\exp\left(-\frac{\alpha}{u_0\tau_0}\right)\right]$$
$$-\varepsilon\sigma\frac{h_{\rho0}}{h_{q0}}\frac{\rho_0M_{f0}^2}{\lambda_0}\left[1-\exp\left(-\frac{\alpha}{u_0\tau_0}\right)\right]+\dots,$$

$$u(\alpha,\beta)=u_0+\varepsilon\left(-\frac{u_0}{\lambda_0}\right)+\varepsilon^2\left(-\frac{p_2}{\rho_0u_0}\right)+\sigma\varepsilon\left(-\frac{\bar{p}}{\rho_0u_0}\right)+\dots, \tag{4.9}$$

$$\rho(\alpha,\beta)=\rho_0+\varepsilon\frac{M_{f0}^2\rho_0}{\lambda_0}+\varepsilon^2\left(\frac{p_2}{a_{f0}^2}-\frac{h_{q0}}{h_{\rho 0}}q_2\right)$$
$$+\varepsilon\sigma\left(\frac{\bar{p}}{a_{f0}^2}-\frac{h_{q0}}{h_{\rho 0}}\bar{q}\right)+\ldots,$$

$$y(\alpha,\beta)=\beta+\varepsilon\alpha+\varepsilon^2\tfrac{1}{2}\frac{h_{q0}}{h_{\rho 0}}\frac{\rho_0 u_0^4}{\lambda_0^2}\eta\beta\left[\exp\left(-\frac{\alpha}{u_0\tau_0}\right)-1\right]$$
$$+\varepsilon\sigma\frac{M_{f0}^2}{2\lambda_0}\beta\exp\left(-\frac{\alpha}{u_0\tau_0}\right)+\ldots,$$

$$x(\alpha,\beta)=\alpha_0+\lambda_0\beta+\varepsilon[-M_{f0}^2\beta-\kappa_1 M_{f0}^4\rho_0 u_0\beta/(2\lambda_0^2)]$$
$$+\varepsilon^2\left\{\lambda_2 y_2+\frac{M_{f0}^2}{\lambda_0}\frac{u_2}{u_0}\beta+\frac{M_{f0}^2}{4}\frac{h_{q0}}{h_{\rho 0}}\frac{\rho_0 u_0^4}{u_0\tau_0\lambda_0^2}\eta\beta^2\exp\left(-\frac{\alpha}{u_0\tau_0}\right)\right.$$
$$+\kappa_1\frac{h_{q0}}{h_{\rho 0}}\frac{M_{f0}^2\rho_0^2 u_0^4}{4\lambda_0^4\tau_0}\eta\beta^2\exp\left(-\frac{\alpha}{u_0\tau_0}\right)+\kappa_2\frac{M_{f0}^4\rho_0^2u_0^2}{2\lambda_0^3}\beta\left[1-\exp\left(-\frac{\alpha}{u_0\tau_0}\right)\right]$$
$$+\varepsilon\sigma\left\{\lambda_0\bar{y}+\tfrac{1}{4}\frac{M_{f0}^4}{\lambda_0\tau_0 u_0}\beta^2\exp\left(-\frac{\alpha}{u_0\tau_0}\right)+\frac{\kappa_1}{4}\frac{M_{f0}^4}{\lambda_0^3\tau_0 u_0}\beta^2\exp\left(-\frac{\alpha}{u_0\tau_0}\right)\right.$$
$$\left.+\kappa_2\frac{h_{\rho 0}}{h_{q0}}\frac{\rho_0 M_{f0}^6}{2\lambda_0^2 u_0^2}\beta\left[\exp\left(-\frac{\alpha}{u_0\tau_0}\right)-1\right]\right\}+\ldots. \quad (4.9)$$

5. Construction of the Front Frozen Shock Wave

Applying the perturbation analysis to the shock wave jump condition (2.7), the so-called "consistency relationship" [see Appendix B] may be easily obtained:

$$\frac{dx}{dy}=\cot\delta=\lambda_0+\varepsilon(-M_{f0}^2\delta_1)+\varepsilon^2(-M_{f0}^2\delta_2)+\varepsilon\sigma(-M_{f0}^2\bar{\delta})+\ldots, \quad (5.1)$$

where

$$\delta_1=\frac{A+1}{4}\frac{M_{f0}^2}{\lambda_0^2}\frac{v_1}{u_0},\qquad \delta_2=\frac{A+1}{4}\frac{M_{f0}^2}{\lambda_0^2}\frac{v_2}{u_0},$$

$$\bar{\delta}=\frac{A+1}{4}\frac{M_{f0}^2}{\lambda_0^2}\frac{\bar{v}}{u_0}\quad\text{and} \quad (5.2)$$

$$A=1+\frac{a_{f0}^2}{\rho_0 h_{\rho 0}}\left[1+\rho^2 a_f^2\left(h_{pp}+\frac{2}{a_f^2}h_{pp}+\frac{1}{a_f^4}h_{\rho\rho}\right)\right]_0.$$

Also, the shock expression is assumed to be

$$\alpha=\varepsilon f_1(\beta)+\varepsilon^2 f_2(\beta)+\varepsilon\sigma\bar{f}(\beta)+\ldots, \quad (3.6)$$

therefore,

$$\cot\delta=\left(\frac{dx}{dy}\right)_{\text{shock}}=\left(\frac{x_\alpha d\alpha+x_\beta d\beta}{y_\alpha d\alpha+y_\beta d\beta}\right)_{\text{shock}}=x_\alpha\frac{d\alpha}{d\beta}+x_\beta$$

where $x(\alpha\beta)$ may be expanded in the following series:

$$x(\alpha,\beta)=x_0(o,\beta)+\varepsilon\left[x_1(o,\beta)+\frac{\partial x_0}{\partial\alpha}\bigg|_{\alpha=0}f_1(\beta)\right]$$

$$+\varepsilon^2\left[x_2(o,\beta)+\frac{\partial x_1}{\partial\alpha}\bigg|_{\alpha=0}f_1(\beta)+\frac{\partial x_0}{\partial\alpha}\bigg|_{\alpha=0}f_2(\beta)\right.$$

$$\left.+\varepsilon\sigma\left[\frac{\partial x_0}{\partial\alpha}\bigg|_{\alpha=0}\bar{f}(\beta)+\frac{\partial x_1}{\partial\alpha}\bigg|_{\alpha=0}f_1(\beta)\right]+\dots,\right. \tag{5.3}$$

and $y=\beta$, at the shock [eq. $(2.6)_3$]. (5.4)

From Eqs. (5.1) – (5.4) and second-order solutions (4.9), one may obtain the following relations:

$$f_1(\beta)=-\left[\frac{A+1}{4}\frac{M_{f0}^4}{\lambda_0^2}-M_{f0}^2-\kappa_1 M_{f0}^4\rho_0 u_0/(2\lambda_0^2)\right]\beta$$

$$f_2(\beta)=\left(\frac{A+1}{4}\frac{M_{f0}^2}{\lambda_0^2}\right)\left(\frac{\beta}{\lambda_0}+\tfrac{1}{4}\frac{h_{q0}}{h_{\rho 0}}\frac{\rho_0 u_0^4}{\tau_0\lambda_0^2}\eta\beta^2\right)$$

$$-\frac{M_{f0}^2}{\lambda_0}\frac{u_2}{u_0}\beta-\frac{M_{f0}^2}{4}\frac{h_{q0}}{h_{\rho 0}}\frac{p_0 u_0^4}{u_0\tau_0\lambda_0^2}\eta\beta^2$$

$$\bar{f}(\beta)=\frac{A+1}{16}\frac{M_{f0}^4}{\lambda_0^3 u_0\tau_0}\beta^2$$

Finally, the second-order front frozen shock is given by

$$\alpha=-\varepsilon\left[\frac{A+1}{4}\frac{M_{f0}^4}{\lambda_2}-M_{f0}^2-\frac{\kappa_1 M_{f0}^4\rho_0 u_0}{2\lambda_0^2}\right]\beta$$

$$+\varepsilon^2\frac{A+1}{4}\frac{M_{f0}^2}{\lambda_0^2}\left(\frac{\beta}{\lambda_0}+\tfrac{1}{4}\frac{h_{q0}}{h_{\rho 0}}\frac{\rho_0 u_0^4}{\tau_0\lambda_0^2}\eta\beta^2\right)-\varepsilon^2\frac{M_{f0}^2}{\lambda_0}\frac{u_2}{u_0}\beta$$

$$-\varepsilon^2\frac{M_{f0}^2}{4}\frac{h_{q0}}{h_{\rho 0}}\frac{\rho_0 u_0^4}{u_0\tau_0\lambda_0^2}\eta\beta^2+\varepsilon\sigma\frac{A+1}{4}\frac{M_{f0}^4}{\lambda_0^3 u_0\tau_0}\beta^2 .$$

It is obvious that the second-order theory presented here is only a near-field nonlinear solution of the problem. Since the products $\varepsilon\beta$, $\varepsilon^2\beta$, $\varepsilon\sigma\beta$, ... that appear in the various terms of the theory become ambiguous as the value of β increases. However, the systematic perturbation calculation of the near-field region can be carried out to any degree of accuracy.

Appendix A

A.1. *Solutions of a Boundary Value Problem*

Systems (4.2), (4.5) and (4.6) all can be solved by the standard Laplace transform technique. For example, by denoting the Laplace transform of a function $w(\alpha,\beta)$ by $\tilde{w}(s,\beta)$ or simply by $\tilde{w}$, i.e.,

$$\tilde{w}=\int_0^\infty w(\alpha,\beta)e^{-\alpha s}d\alpha\,, \tag{A.1}$$

then system (4.5) may be reduced to

$$\frac{d^2\tilde{v}_2}{d\beta^2}-2\lambda_0 s\frac{d\tilde{v}_2}{d\beta}=\frac{h_{q0}}{h_{\rho 0}}\frac{\rho_0 u_0^4}{\lambda_0\tau_0}\frac{\eta s}{\left(s+\dfrac{1}{u_0\tau_0}\right)}, \tag{A.2}$$

and the corresponding boundary conditions for $\tilde{v}_2$ are $\tilde{v}_2 = u_1/s$, at $\beta = 0$; and $\tilde{v}_2$ is bounded when $\beta \to \infty$.

The solution for this boundary value problem can be found to be

$$\tilde{v}_2(s, \beta) = \frac{u_1}{s} - \frac{1}{2}\frac{(h_{q0})}{(h_{\rho 0})}\frac{\rho_0 u_0^3}{\tau_0 \lambda_0^2}\frac{\eta\beta}{s + \frac{1}{u_0 \tau_0}}, \tag{A.3}$$

where η and τ_0 are defined in Eq. (3.3).

Equation (A.3) can be readily inverted back to the α, β-plane as:

$$v_2(\alpha, \beta) = -\frac{u_0}{\lambda_0} - \frac{1}{2}\frac{h_{q0}}{h_{\rho 0}}\frac{\rho_0 u_0^4 \eta\beta}{\tau_0 \lambda_0^2}\exp\left(-\frac{\alpha}{u_0 \tau_0}\right). \tag{A.4}$$

Similarly, one can find the solutions for p_2, q_2, etc.

Appendix B

B.1. *Consistent Relationship at the Front Shock*

The shock angle δ may be expanded as follows:

$$\delta(\alpha, \beta) = \delta_0 + \varepsilon\delta_1(\alpha, \beta) + \varepsilon^2\delta_2(\alpha, \beta) + \varepsilon\sigma\bar{\delta}(\alpha, \beta) + \ldots, \tag{B.1}$$

where $\delta_0 = \mu_{f0}$, the free stream frozen Mach angle.

Applying the perturbation scheme to the jump conditions of the shock wave [Eq. (2.7)], the ε-order boundary conditions at the shock are

$$\begin{aligned} &\rho_0(u_1 - v_1 \cot\delta_0) + \rho_1 u_0 = 0, \quad && h_{p0} p_1 + h_{\rho 0}\rho_1 + h_{q0} q_1 + u_0 u_1 = 0, \\ &q_1 = 0,\ p_1 + \rho_0 u_0 u_1 = 0, \quad && v_1 + u_1 \cot\delta_0 = 0. \end{aligned} \tag{B.2}$$

It is noted that Eq. (B.2) is a homogeneous system in p_1, ρ_1, u_1, v_1 and q_1. Likewise, collecting terms of ε^2, one obtains

$$\rho_0(u_2 - v_2 \cot\delta_0) + \rho_2 u_0 = -\rho_0 M_{f0}^2 \delta_1 v_1 + \rho_0 u_0 \frac{M_{f0}^2}{\lambda_0^2}\frac{v_1^2}{a_{f0}^2}, \tag{B.3}$$

$$h_{p0} p_2 + h_{\rho 0}\rho_2 + h_{q0} q_2 + u_2 = -\frac{1}{2}\frac{M_{f0}^2}{\lambda_0^2} v_1 \left\{1 + \rho_0^2 a_{f0}^2 \left(h_{pp} + \frac{2}{a_f^2} h_{p\rho} + \frac{1}{a_f^4} h_{\rho\rho}\right)_0\right\},$$

$$q_2 = 0, \quad p_2 + \rho_0 u_0 u_2 = 0, \quad v_2 + u_2 \cot\delta_0 = -M_{f0}^2 \delta_1 \frac{v_1}{\lambda_0}.$$

It is also noted that the homogeneous parts of Eqs. (B.3) and (B.2) are the same. In particular, their determinants are zero when $\delta_0 = \mu_{f0}$. Consequently, to ensure the system (B.3) is consistent, its nonhomogeneous terms must be related appropriately. This consistent relationship may be easily obtained by eliminating the second order quantities from the system (B.3). One has

$$\delta_1 = \frac{A+1}{4}\frac{M_{f0}^2}{\lambda_0}\frac{v_1}{u_0}, \tag{B.4}$$

where A is defined in Eq. (5.2).

Details of the above derivation can be seen in author's earlier work [9].

Carrying the analysis to higher order systems, i.e., $O(\varepsilon^4)$ and $O(\varepsilon^2\sigma^2)$, one may obtain the expressions for δ_2 and $\bar{\delta}$ as given in Eq. (5.2). The tedious derivation for the higher-order systems is omitted here.

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